Simple Drawings of $K_{m,n}$ Contain Shooting Stars*

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Abstract

Simple drawings are drawings of graphs in which all edges have at most one common point (either a common endpoint, or a proper crossing). It has been an open question whether every simple drawing of a complete bipartite graph $K_{m,n}$ contains a plane spanning tree as a subdrawing. We answer this question to the positive by showing that for every simple drawing of $K_{m,n}$ and for every vertex v in that drawing, the drawing contains a *shooting star* rooted at v, that is, a plane spanning tree with all incident edges of v.

1 Introduction

A *simple drawing* is a drawing of a graph on the sphere S^2 or, equivalently, in the Euclidean plane where (1) the vertices are distinct points in the plane, (2) the edges are non-self-intersecting continuous curves connecting their incident points, (3) no edge passes through vertices other than its incident vertices, (4) and every pair of edges intersects at most once, either in a common endpoint, or in the relative interior of both edges, forming a proper crossing. Simple drawings are also called *good drawings* [3, 5] or *(simple) topological graphs* [7, 8]. In *semi-simple drawings*, the last requirement is softened such that edges without common endpoints are allowed to cross several times. Note that in any simple or semi-simple drawing, there are no tangencies between edges and incident edges do not cross. If a drawing does not contain any crossing at all, it is called *plane*.

The search for plane subdrawings of a given drawing has been a widely considered topic for simple drawings of the complete graph K_n which still holds tantalizing open problems. For example, Rafla [10] conjectured that every simple drawing of K_n contains a plane Hamiltonian cycle, a statement which is by now known to be true for $n \leq 9$ [1] and several classes of simple drawings (e.g., 2-page book drawings, monotone drawings, cylindrical drawings), but still remains open in general. A related question concerns the least number of pairwise disjoint edges in any simple drawing of K_n. The currently best lower bound of $\Omega(n^{1/2-\varepsilon})$, for any $\varepsilon > 0$, for this number has been obtained by Ruiz-Vargas [11], improving over several previous bounds [8, 9, 12], while the trivial upper bound of $n/2$ would be implied by a positive answer to Rafla's conjecture. A structural result of Fulek and Ruiz-Vargas [6] implies that every simple drawing of K_n contains a plane sub-drawing with at least $2n - 3$ edges.

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In this work, we consider the search for plane spanning trees in drawings of complete bipartite graphs. For complete graphs, plane spanning trees trivially exist in both simple and semi-simple drawings, as incident edges don't cross, and as every vertex is incident to all other vertices. Hence the *star* of any vertex, which consists of all edges incident to that vertex and all vertices, is a plane spanning tree.

The task of finding plane spanning trees in drawings of complete bipartite graphs $K_{m,n}$ turns out to be more involved. In fact, not every semi-simple drawing of a complete bipartite graph contains a plane spanning tree; see Figure 1.

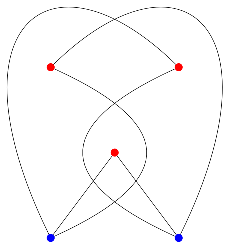

Figure 1 A semi-simple drawing of $K_{2,3}$ that does not contain a plane spanning tree.

For simple drawings of $K_{m,n}$, the existence of plane spanning trees has so far only been proven for specific types of drawings. It is not too hard to see that monotone drawings always contain plane spanning trees. Aichholzer et al. showed in [2] that simple drawings of $K_{2,n}$ and $K_{3,n}$, as well as so-called outer drawings of $K_{m,n}$ [4], always contain plane spanning trees of a special type, which they called shooting stars. A *shooting star rooted at v* is a plane spanning tree with root v that has height 2 and contains the star of vertex v.

In the present work, we show that every simple drawing of $K_{m,n}$ contains shooting stars rooted at an arbitrary vertex of $K_{m,n}$.

▶ **Theorem 1.1.** *Let D be a simple drawing of the complete bipartite graph $K_{m,n}$ and let r be an arbitrary vertex of $K_{m,n}$. Then D contains a shooting star rooted at r.*

2 Proof of Theorem 1.1

Proof. We can assume that D is drawn on a point set $S = R \cup B$, $R = \{r_1, \ldots, r_m\}$, $B = \{b_1, \ldots, b_n\}$, in which the points in the two vertex partitions R and B are colored red and blue, respectively. Without loss of generality let $r = r_1$.

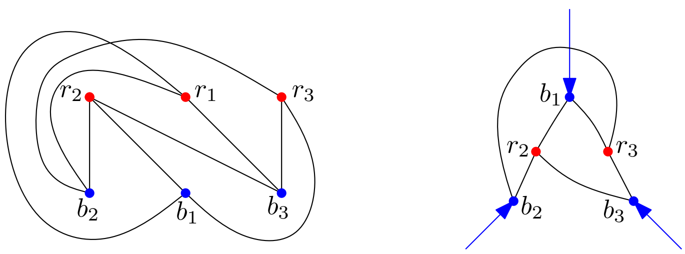

Figure 2 A simple drawing of $K_{3,3}$ (left) and its stereographic projection from r_1 (right).

To simplify the figures, we consider the drawing D on the sphere and apply a stereographic projection from r onto a plane. In that way, the edges in the star of r are represented as (not necessarily straight-line) infinite rays; see Figure 2. We will depict them in blue. In the following, we consider all edges oriented from their red to their blue endpoint. In order to specify how two edges cross each other, we introduce some notation. Consider two crossing edges $e_1 = r_i b_k$ and $e_2 = r_j b_l$ and let $\times$ be their crossing point. Consider the arcs $\times r_i$ and $\times b_k$ on e_1 and $\times r_j$ and $\times b_l$ on e_2. We say that e_2 crosses e_1 in *clockwise direction* if the clockwise cyclic order of these arcs around the crossing $\times$ is $\times r_i$, $\times r_j$, $\times b_k$, and $\times b_l$. Otherwise, we say that e_2 crosses e_1 in *counterclockwise direction*; see Figure 3.

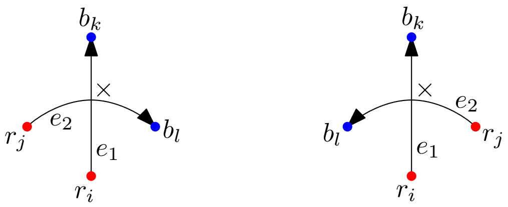

Figure 3 Edge e_2 crosses edge e_1 in clockwise direction (left) or counterclockwise direction (right).

We prove Theorem 1.1 by induction on n. For $n = 1$ and any $m \geq 1$, the whole drawing D is a shooting star rooted at any vertex, and in particular at r.

Assume that the existence of shooting stars rooted at any vertex has been proven for any simple drawing of $K_{m,n'}$ with $n' < n$. Let M be a subset of the edges of D connecting each vertex $r_i \neq r$ to some blue vertex in B (with exactly one edge for every red vertex), such that (i) $M \cup \{\bigcup_{j=2}^{n} rb_j\}$ does not contain any crossing and (ii) the number of crossings of M with edge rb_1 is the minimum possible. Observe that the set M is well defined, since, by the induction hypothesis, the subdrawing of D obtained by deleting the blue vertex b_1 and its incident edges contains a shooting star rooted at r. Thus there exists a set M_1 of edges from D connecting each $r_i \neq r$ to some blue vertex in $B \setminus \{b_1\}$ such that $M_1 \cup \{\bigcup_{j=2}^{n} rb_j\}$ does not contain any crossing. As arbitrarily many of the edges in M_1 might cross rb_1, M_1 might not fulfill condition (ii). We will show that M does not contain any crossing with rb_1. Since $M \cup \{\bigcup_{j=2}^{n} rb_j\}$ does not contain crossings by construction and as rb_1 does not cross any edges of $\{\bigcup_{j=2}^{n} rb_j\}$ in a simple drawing, it follows that the edges in $M \cup \{\bigcup_{j=1}^{n} rb_j\}$ form the desired shooting star.

Assume for a contradiction that rb_1 crosses at least one edge in M. When traversing rb_1 from b_1 to r, let x be the first crossing point of rb_1 with an edge $r_k b_t$ in M. Without loss of generality, when orienting rb_1 from r to b_1 and $r_k b_t$ from r_k to b_t, $r_k b_t$ crosses rb_1 in counterclockwise direction (otherwise we can mirror the drawing).

Suppose first that the arc $r_k x$ (on $r_k b_t$ and oriented from r_k to x) is crossed in counterclockwise direction by an edge incident to b_1 (and oriented from the red endpoint to b_1). Let $e = r_l b_1$ be such an edge whose crossing with $r_k x$ at a point y is the closest to x. Otherwise, let e be the edge $r_k b_1$ and y be the point r_k. In the remaining figures, we represent in blue the edges of the star of r, in red the edges in M, and in black the edge e.

We distinguish two cases depending on whether e crosses an edge of the star of r. The idea in both cases is to define a region Γ and, inside it, redefine the connections between red and blue points to reach a contradiction.

Case 1: e does not cross any edge of the star of r. Let Γ be the closed region of the plane bounded by the arcs yb_1 (on e), $b_1 x$ (on rb_1), and xy (on $r_k b_t$); see Figure 4. Observe that all the blue points b_j lie outside the region Γ and that for all the red points r_i inside region Γ, the edge $r_i b_1$ must be in Γ. Let M_Γ denote the set of edges $r_i b_1$ with $r_i \in \Gamma$ and note that $r_k b_1 \in M_\Gamma$. Consider the set M' of red edges obtained from M by replacing, for each red point $r_i \in \Gamma$, the (unique) edge incident to r_i in M by the edge $r_i b_1$ in M_Γ, and keeping the

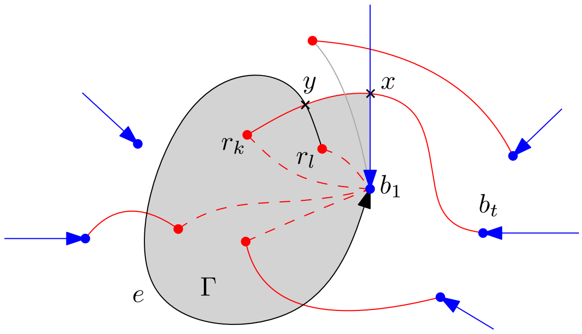

Figure 4 Illustration of Case 1.

other edges in M unchanged. In particular, the edge r_kb_t has been replaced by the edge r_kb_1. The edges in M_Γ neither cross each other nor cross any of the blue edges rb_j. Moreover, we now show that the non-replaced edges in M must lie completely outside Γ. These edges can neither cross r_kb_t (by definition of M) nor the arc b_1x (on rb_1). Thus, if they are incident to b_1 they cannot cross the boundary of Γ, and otherwise their endpoints lie outside Γ and they can only cross one arc of the boundary. Therefore, M' satisfies that $M' \cup \{\bigcup_{j=2}^{n} rb_j\}$ does not contain any crossing, and has fewer crossings with rb_1 than M. This contradicts the definition of M as the one with the minimum number of crossings with rb_1.

Case 2: e crosses the star of r. When traversing e from r_k or r_l (depending on the definition of e) to b_1, let $I = \{\alpha, \beta, \ldots, \rho\}$ be the indices of the edges of the star of r in the order as they are crossed by e and let $y_\alpha, \ldots, y_\rho$ be the corresponding crossing points on e. Note that, when orienting e from r_k or r_l to b_1, the edges $rb_\xi, \xi \in I$, oriented from r to b_ξ, cross e in counterclockwise direction, since they can neither cross r_kb_t (by definition of M) nor rb_1.

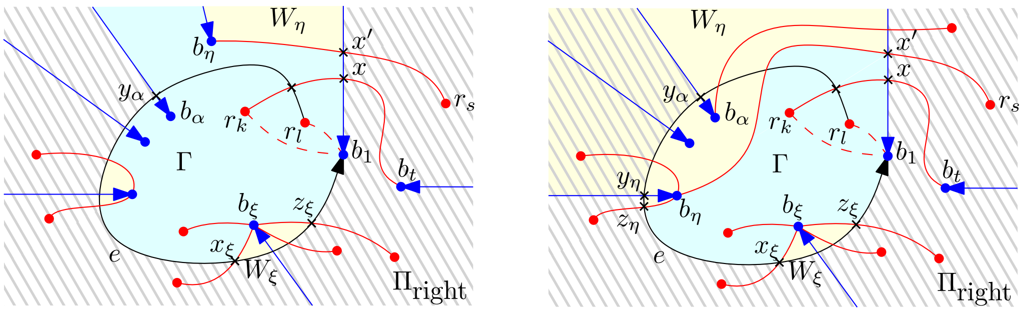

Figure 5 Illustration of Case 2. Region Π_{right} is striped in gray, region Γ is shaded in blue, and regions in $\bigcup_{\xi \in I} W_\xi \cup W_\eta$ are shaded in yellow. Left: $\eta \notin I$. Right: $\eta \in I$.

The three arcs ry_α (on rb_α), $y_\alpha b_1$ (on e), and b_1r divide the plane into two (closed) regions, Π_{left}, containing vertex r_k, and Π_{right}, containing vertex b_t. For each $\xi \in I$, let M_ξ be the set of red edges of M incident to some red point in Π_{right} and to b_ξ. Note that all the edges in M_ξ (if any) must cross the edge e. When traversing e from r_k or r_l to b_1, we denote by x_ξ, z_ξ the first and the last crossing points of e with the edges of $M_\xi \cup rb_\xi$, respectively; see Figure 5 for an illustration. We remark that both x_ξ and z_ξ might coincide with y_ξ and, in particular, if $M_\xi = \emptyset$ then $x_\xi = y_\xi = z_\xi$.

We now define some regions in the drawing D. Suppose first that there are edges in M (oriented from the red to the blue point) that cross rb_1 (oriented from r to b_1) in clockwise direction. Let r_sb_η be the edge in M whose clockwise crossing with rb_1 at a point x' is the closest one to x (recall that the arc b_1x on rb_1 is not crossed by edges in M). Then, if $\eta \notin I$, we denote by W_η the region bounded by the arcs rx' (on rb_1), $x'b_\eta$ (on r_sb_η), and rb_η and not containing b_1; see Figure 5 (left). If $\eta \in I$, we define W_η as the region bounded by the arcs rx' (on rb_1), $x'b_\eta$ (on r_sb_η), $b_\eta z_\eta$, $z_\eta y_\eta$ (on e), and $y_\eta r$ (on rb_η) and not containing b_1; see Figure 5 (right). If no edges in M cross rb_1 in clockwise direction, we set $W_\eta = \emptyset$. Moreover, for each $\xi \in I \setminus \{\eta\}$, we define W_ξ as the region bounded by the arcs $x_\xi b_\xi$, $b_\xi z_\xi$, and $z_\xi x_\xi$ (and not containing b_1); see again Figure 5.

We can finally define the region Γ for Case 2, which is the region obtained from Π_{left} by removing the interior of all the regions W_ξ, $\xi \in I$ plus region W_η if $\eta \notin I$ (otherwise it is already contained in $\bigcup_{\xi \in I} W_\xi$). Now consider the set of red and blue vertices contained in the region Γ. Let J denote the set of indices such that for all $j \in J$, the blue point b_j lies in Γ (note that $1 \in J$). Since b_t is not in Γ, by the induction hypothesis, we can find a set of edges M_Γ connecting each red point in Γ with a blue point b_j, $j \in J$ satisfying that $M_\Gamma \cup \{\bigcup_{j \in J} rb_j\}$ does not contain any crossing. Moreover, all the edges in M_Γ lie entirely in Γ: An edge in M_Γ cannot cross any of the edges rb_j, with $j \in J$. Thus, it cannot leave Π_{left}, as otherwise it would cross e twice. Further, if it entered one of the regions in $\bigcup_{\xi \in I} W_\xi \cup W_\eta$, it would have to leave it crossing e, and then it could not reenter Γ.

Consider the set M' of red edges obtained from M by replacing, for each red point $r_i \in \Gamma$, the edge r_ib_ξ in M by the edge r_ib_j, $j \in J$, in M_Γ, and keeping the other edges in M unchanged. In particular, the edge r_kb_t has been replaced by some edge r_kb_j, $j \in J$. The edges in M_Γ neither cross each other nor cross any of the blue edges rb_j, $j \in J$ nor any of the other ones, lying completely outside Γ. Moreover, the non-replaced edges in M cannot enter Γ since the only boundary part of Γ that they can cross are arcs on e. Therefore, M' satisfies that $M' \cup \{\bigcup_{j=2}^{n} rb_j\}$ does not contain any crossing, and has fewer crossings with rb_1 than M. This contradicts the definition of M as the one with the minimum number of crossings with rb_1. ◀

3 Some Observations on Tightness

There exist simple drawings of $K_{m,n}$ in which every plane subdrawing has at most as many edges as a shooting star. For example, consider a straight line drawing of $K_{m,n}$ where all vertices are in convex position such that all red points are next to each other in the convex hull; see Figure 6 (left). The convex hull is a $(m+n)$-gon which shares only two edges with the drawing of $K_{m,n}$; see Figure 6 (right). All other edges of the drawing of $K_{m,n}$ are diagonals of the polygon. As there can be at most $(m+n)-3$ pairwise non-crossing diagonals in a convex $(m+n)$-gon, any plane subdrawing of this drawing of $K_{m,n}$ contains at most $m+n-1$ edges.

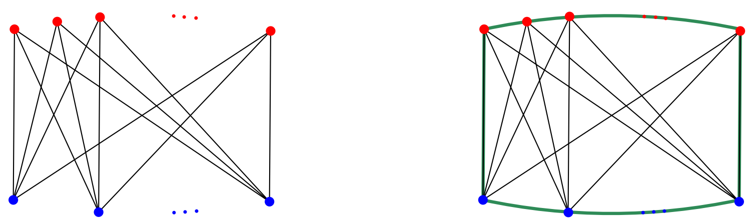

Figure 6 Left: A simple drawing of $K_{m,n}$ where no plane subdrawing can have more edges than a shooting star. Right: A convex $(n+m)$-gon (in green lines) around the simple drawing of $K_{m,n}$.

Furthermore, both requirements from Theorem 1.1 – simplicity of the drawing and having a complete bipartite graph – are in fact necessary: As mentioned in the introduction, not all semi-simple drawings of $K_{m,n}$ contain a plane spanning tree. Further, if in the example in Figure 6, we delete one of the two edges of $K_{m,n}$ on the boundary of the convex hull, then any plane subdrawing has at most $m+n-2$ edges. Hence the resulting drawing cannot contain any plane spanning tree.

4 Remarks on an Algorithm

The proof of Theorem 1.1 contains an algorithm with which we can find shooting stars in given simple drawings. We start with constructing the shooting star for a subdrawing that is a $K_{m,1}$ and then inductively add more vertices. Every time we are adding a new vertex, the shooting star of the step before is a set fulfilling all requirements of $M_1 \cup \{\bigcup_{j=2}^{n} rb_j\}$ in the proof. By replacing edges as described in the proof, we obtain a new set with the same properties and fewer crossings. We continue replacing edges until we obtain a set of edges (M in the proof) that form a shooting star for the extended vertex set. We remark that the runtime of this algorithm might be exponential, as finding the edges of M_Γ might require solving the problem for the subgraph induced by Γ. However, we believe that there exists a polynomial-time algorithm for this task.

▶ Open Problem 1. Given a simple drawing of the complete bipartite graph, is there a polynomial-time algorithm to find a plane spanning tree contained in the drawing?

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