Rotational symmetric flexible placements of graphs*

Sean Dewar[1], Georg Grasegger[1], and Jan Legerský[2,3]

1 **Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences**
sean.dewar@ricam.oeaw.ac.at, georg.grasegger@ricam.oeaw.ac.at

2 **Johannes Kepler University Linz, Research Institute for Symbolic Computation (RISC)**
jan.legersky@risc.jku.at

3 **Department of Applied Mathematics, Faculty of Information Technology, Czech Technical University in Prague**

Abstract

We study the existence of an n-fold rotational symmetric placement of a symmetric graph in the plane allowing a continuous deformation that preserves the symmetry and the distances between adjacent vertices. We show that such a flexible placement exists if and only if the graph has a NAC-colouring satisfying an additional property on the symmetry; a NAC-colouring is a surjective edge colouring by two colours such that every cycle is either monochromatic, or there are at least two edges of each colour.

Rigid graphs are those which have only finitely many non-congruent placements in the plane with the same edge lengths as a generic placement. These graphs can, however, have non-generic special choices of a placement such that there are infinitely many non-congruent placements in the plane with the same edge lengths. We call such a placement *flexible*.

The study of flexible placements of generically rigid graphs has a long history. Dixon found two types of flexible placements of the bipartite graph $K_{3,3}$ [3, 17, 14]. Walter and Husty [15] proved that these are indeed all (assuming that vertices do not overlap). Figure 1 shows some special symmetric cases of these two constructions applied to $K_{4,4}$. Further examples of graphs with flexible placements are Burmester's focal point mechanism [1], a 12-vertex graph studied by Kempe [11], and two constructions by Wunderlich [16, 18].

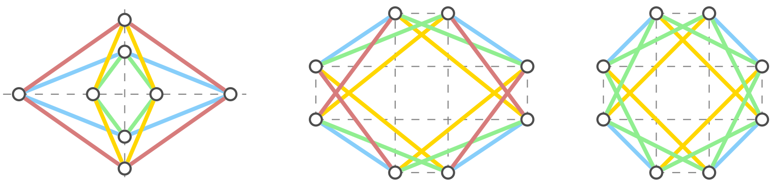

Figure 1 The vertices of $K_{4,4}$ can be placed symmetrically on orthogonal lines to make the graph flexible with 2-fold rotational symmetry (left). A 2-fold rotationally symmetric flexible instance of $K_{4,4}$ is obtained by placing the vertices of each part to a rectangle so that the two rectangles have the same intersection of diagonals and parallel/orthogonal edges (middle). Although there is a 4-fold rotationally symmetric choice of rectangles (right), the deformed placements preserving the edge lengths are only 2-fold symmetric. The colours indicate equality of edge lengths in a placement.

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In recent works [7, 8, 4] a deeper analysis of existence of flexible placements is done via graph colourings. There is a special type of edge colourings, called NAC-colourings ("No Almost Cycles", see [7]), which classify the existence of a flexible placement in the plane and give a construction of the motion. Furthermore, determining the NAC-colourings of a given graph and the possible constructions that come from it can be done easily by using the SageMath package FlexRiLoG [5]. In [6] we used these methods for constructing flexible placements for symmetric graphs as in Figure 2. However, we did not take advantage of the symmetry for the construction, and instead had to construct the framework manually.

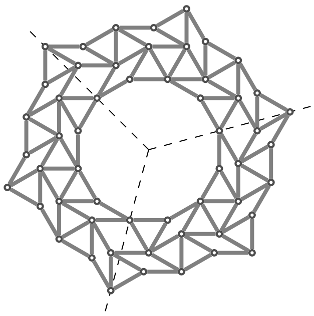

Figure 2 A symmetric graph which has a 3-fold rotationally symmetric flexible placement.

Symmetry plays an important role in art and design, and often appears in nature also. Due to this, there is a large body of work focused on symmetric frameworks and their properties in the context of rigidity theory; see [10, 12]. In particular, we shall be focusing on graphs and frameworks that display n-fold rotational symmetry, such as in Figure 2.

In this extended abstract, we formalise the NAC-colouring method for rotationally symmetric flexible placements, i. e. in such a way that the motion preserves the symmetry. By combining Theorem 3.1 and Theorem 3.2, we obtain the following result:

Let G be a $\mathcal{C}_n$-symmetric connected graph. Then G has a $\mathcal{C}_n$-symmetric NAC-colouring if and only if there exists a $\mathcal{C}_n$-symmetric flexible framework (G, p) in $\mathbb{R}^2$.

1 Preliminaries

We briefly recall some basic notions from rigidity theory and define NAC-colourings in this section. All graphs $G = (V(G), E(G))$ in the paper are connected and $|E(G)| \geq 1$.

▶ **Definition 1.1.** A *framework in* $\mathbb{R}^2$ is a pair (G, p) where G is a (finite simple) graph and $p : V(G) \to \mathbb{R}^2$ is a *placement* of G, a possibly non-injective map such $p(u) \neq p(v)$ if $uv \in E(G)$. We define frameworks (G, p) and (G, q) to be *equivalent* if for all $uv \in E(G)$,

$$\|p(u) - p(v)\| = \|q(u) - q(v)\| . \tag{1}$$

We define two placements p, q of G to be *congruent* if (1) holds for all $u, v \in V(G)$.

An equivalent definition for congruence is as follows; p and q are congruent if there exists an Euclidean isometry M of $\mathbb{R}^2$ such that $Mq(v) = p(v)$ for all $v \in V(G)$.

▶ **Definition 1.2.** Let (G, p) be a framework. A *flex (in* $\mathbb{R}^2$*)* of (G, p) is a continuous path $t \mapsto p_t$, $t \in [0, 1]$, in the space of placements of G such that $p_0 = p$ and each (G, p_t) is equivalent to (G, p). If p_t is congruent to p for all $t \in [0, 1]$ then p_t is *trivial*. We define (G, p) to be *flexible* if there is a non-trivial flex of (G, p) in $\mathbb{R}^2$, and *rigid* otherwise.

It was shown in [13] that a framework (G,p) with a generic placement of vertices (see [9]) is rigid if and only if G contains a Laman graph as a spanning subgraph. This does not inform us whether a graph will have a flexible placement; for example, any generic placement of $K_{4,4}$ is rigid, however as shown by Figure 1 we can construct flexible placements for it. To determine whether a graph has flexible placements we introduce the following.

▶ **Definition 1.3.** An edge colouring $\delta : E(G) \to \{\text{red}, \text{blue}\}$ of a graph G is a *NAC-colouring* if $\delta(E(G)) = \{\text{red}, \text{blue}\}$ and for each cycle in G, either all edges have the same colour, or there are at least two red and two blue edges. NAC-colourings $\delta, \overline{\delta}$ of G are *conjugated* if $\delta(e) \neq \overline{\delta}(e)$ for all $e \in E(G)$.

▶ Remark. The colourings considered within this paper are not required to have incident edges coloured differently, contrary to the common graph-theoretical terminology.

Having these definitions, we can recall the result [7, Theorem 3.1].

▶ **Theorem 1.4.** *A graph has a flexible placement in $\mathbb{R}^2$ if and only if it has a NAC-colouring.*

2 Rotational symmetry

The following definitions specify the rotational symmetric setting where Definition 2.1 gives a combinatorial description of symmetry of a graph and Definition 2.2 describes geometric symmetry of a framework. Note, that in figures we describe vertices in graphs by filled disks and in frameworks with circles.

▶ **Definition 2.1.** Let G be a graph and $n \geq 2$. Let the *n-fold rotation group*, $\mathcal{C}_n := \langle \omega : \omega^n = 1 \rangle$ act on G, i.e., there exists an injective group homomorphism $\theta : \mathcal{C}_n \to \mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ is the automorphism group of G. We define $\gamma v := \theta(\gamma)(v)$ for $\gamma \in \mathcal{C}_n$; similarly, for any edge $e = uv \in E(G)$, we define $\gamma e := \gamma u \gamma v$. We shall define $v \in V(G)$ to be an *invariant vertex* if $\gamma v = v$ for all $\gamma \in \mathcal{C}_n$, and *partially invariant* if $\gamma v = v$ for some $\gamma \in \mathcal{C}_n, \gamma \neq 1$. The graph G is called *$\mathcal{C}_n$-symmetric* if:

(a) a vertex is invariant if and only if it is partially invariant, and
(b) the set of invariant vertices of G forms an independent set.

▶ **Definition 2.2.** Let (G,p) be a framework in $\mathbb{R}^2$, G be $\mathcal{C}_n$-symmetric and $\tau : \mathcal{C}_n \to O(2,\mathbb{R})$ be a *symmetry map*, i.e., an injective group homomorphism, given by

$$\tau(\omega) = \begin{bmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}.$$

If $p(\gamma v) = \tau(\gamma)p(v)$ for each $v \in V(G)$ and $\gamma \in \mathcal{C}_n$, then (G,p) is a *$\mathcal{C}_n$-symmetric framework*; likewise, we define p to be a *$\mathcal{C}_n$-symmetric placement* of G.

We note that if (G,p) is $\mathcal{C}_n$-symmetric then it is $\mathcal{C}_m$-symmetric for all $m|n$.

▶ **Definition 2.3.** Let (G,p) be a $\mathcal{C}_n$-symmetric framework in $\mathbb{R}^2$. If there is a non-trivial flex p_t of (G,p) such that each (G,p_t) is $\mathcal{C}_n$-symmetric, then (G,p) is *$\mathcal{C}_n$-symmetric flexible* (or *n-fold rotation symmetric flexible*), and *$\mathcal{C}_n$-symmetric rigid* otherwise.

We define the following for edge colourings of $\mathcal{C}_n$-symmetric graphs.

▶ **Definition 2.4.** Let G be a $\mathcal{C}_n$-symmetric graph with colouring δ. A *red, resp. blue, component* is a connected component of $G^\delta_{\text{red}} := (V(G), \{e \in E(G) : \delta(e) = \text{red}\})$, resp. G^δ_{blue}. A blue or red component $H \subset G$ is *partially invariant* if there exists $\gamma \in \mathcal{C}_n \setminus \{1\}$ such that $\gamma H = H$, and *invariant* if $\gamma H = H$ for all $\gamma \in \mathcal{C}_n$ (see Figure 3 for an example).

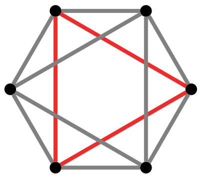

Figure 3 A partially invariant (but not invariant) red component on the left and an invariant red component (and therefore also partially invariant) on the right for $\mathcal{C}_6$-symmetry. The symmetry is indicated by the graph layout.

We focus on the following class of NAC-colourings suitable for dealing with symmetries.

▶ **Definition 2.5.** Let G be a $\mathcal{C}_n$-symmetric graph with NAC-colouring δ. We define δ to be a *$\mathcal{C}_n$-symmetric NAC-colouring* if $\delta(\gamma e) = \delta(e)$ for all $e \in E(G)$ and $\gamma \in \mathcal{C}_n$ and no two distinct blue, resp. red, partially invariant components are connected by an edge (see Figure 4 for examples).

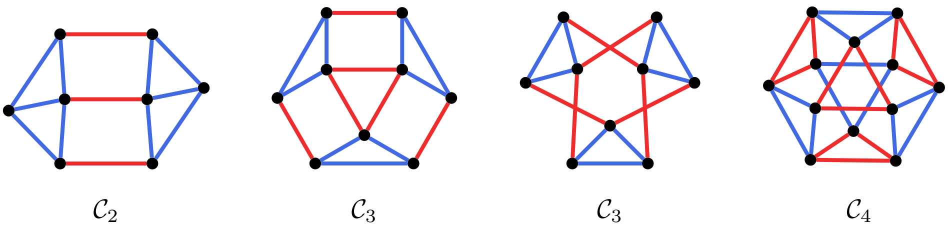

Figure 4 Valid $\mathcal{C}_n$-symmetric NAC-colouring for some graphs. For the first three graphs the symmetry is indicated by the layout. There is a $\mathcal{C}_4$-symmetry for the graph on the right which is not geometrically visible in the figure; see Figure 8 for a $\mathcal{C}_4$-symmetric placement.

▶ **Example 2.6.** The cartesian product of K_4 and K_2 has a single NAC-colouring (up to conjugation) δ, see Figure 5. The graph is $\mathcal{C}_2$-symmetric under the symmetry θ, where $\theta(\omega)$ is the permutation $(1,6)(2,5)(3,8)(4,7)$; further, the NAC-colouring δ is a $\mathcal{C}_2$-symmetric NAC colouring with respect to θ. The graph is also $\mathcal{C}_4$-symmetric under the symmetry θ', where $\theta'(\omega)$ is the permutation $(1,6,3,8)(5,2,7,4)$; however, the NAC-colouring δ is not a $\mathcal{C}_4$-symmetric NAC colouring with respect to θ', since the blue partially invariant components are connected by edges.

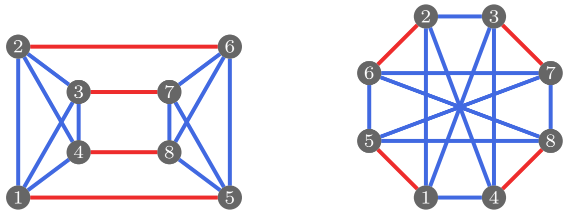

Figure 5 The cartesian product of K_4 and K_2 has only one NAC-colouring. It is $\mathcal{C}_2$-symmetric (left) but not $\mathcal{C}_4$-symmetric (right). The symmetries are indicated by the graph layout.

3 Necessary and sufficient conditions for rotationally symmetric flexibility

In this section we show the construction of $\mathcal{C}_n$-symmetric motions from $\mathcal{C}_n$-symmetric NAC-colourings. The inverse direction is also true.

▶ **Theorem 3.1.** *If (G, p) is a $\mathcal{C}_n$-symmetric flexible framework in $\mathbb{R}^2$, G being a connected graph, then G has a $\mathcal{C}_n$-symmetric NAC-colouring.*

The proof of Theorem 3.1 follows in a similar vein to the proof of [7, Theorem 3.1] by using methods from valuation theory. There are more complexities involved however, since we also require that the colouring obtained respects the symmetry of the graph and partially invariant components are not connected by edges.

▶ **Theorem 3.2.** *Let G be a $\mathcal{C}_n$-symmetric connected graph. If G has a $\mathcal{C}_n$-symmetric NAC-colouring δ, then there exists a $\mathcal{C}_n$-symmetric flexible framework (G, p) in $\mathbb{R}^2$.*

Proof. The proof is based on the "zigzag" grid construction from [7] with a specific choice of the grid. Let $R_1^0, \dots, R_1^{n-1}, \dots, R_m^0, \dots, R_m^{n-1}$ be the red components of G^δ_{red} that are not partially invariant. We can assume that $R_j^i = \omega^i R_j^0$ for $0 \leq i < n$ and $1 \leq j \leq m$. Similarly, let $B_1^0, \dots, B_1^{n-1}, \dots, B_k^0, \dots, B_k^{n-1}$ be the blue components of G^δ_{blue} that are not partially invariant and $B_j^i = \omega^i B_j^0$ for $0 \leq i < n$ and $1 \leq j \leq k$.

Let $a_1, \dots, a_m$ and $b_1, \dots, b_k$ be points in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $a_j \neq \tau(\omega)^i a_{j'}$ and $b_j \neq \tau(\omega)^i b_{j'}$ for $j \neq j'$ and $1 \leq i < n$ arbitrary. We define functions $\overline{a}, \overline{b} \colon V(G) \to \mathbb{R}^2$ by

$$\overline{a}(v) = \begin{cases} \tau(\omega)^i a_j & \text{if } v \in R_j^i \\ (0,0) & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \overline{b}(v) = \begin{cases} \tau(\omega)^i b_j & \text{if } v \in B_j^i \\ (0,0) & \text{otherwise.} \end{cases}$$

We note that a vertex is mapped to the origin by $\overline{a}$ (respectively, $\overline{b}$) if and only if it lies in a red (respectively, blue) partially invariant component. We now obtain for each $t \in [0, 2\pi]$ a placement p_t of G, where

$$p_t(v) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \overline{a}(v) + \overline{b}(v)\,. \tag{2}$$

Let $uv \in E(G)$. If $\delta(uv)$ is red (resp. blue) then $\overline{a}(u) = \overline{a}(v)$ (resp. $\overline{b}(u) = \overline{b}(v)$). Hence, the edge length $\|p_t(u) - p_t(v)\|$ is independent of t.

Next, we have to show that no two adjacent vertices are mapped to the same point by the placement p_0. Assume that $\big(\overline{a}(u), \overline{b}(u)\big) = \big(\overline{a}(v), \overline{b}(v)\big)$ for some vertices u, v. Suppose this is due to the fact that u and v belong to the same red and same blue (possibly partially invariant) component. Hence, $uv \notin E(G)$, otherwise δ is not a NAC-colouring (uv would yield a cycle with a single edge in one color). On the other hand, if u and v are in two different red (resp. blue) components, then $\overline{a}(u) = \overline{a}(v) = (0,0)$ (resp. $\overline{b}(u) = \overline{b}(v) = (0,0)$). By our construction of $\overline{a}$ (resp. $\overline{b}$), it follows that u, v both lie in partially invariant red (resp. blue) components. Since these components are partially invariant, $uv \notin E(G)$ by the assumption that δ is $\mathcal{C}_n$-symmetric.

As edge lengths are preserved and no two vertices connected by an edge are mapped to the same point, $(G, p) := (G, p_0)$ is a framework with a flex p_t. Further, the flex is not trivial by surjectivity of δ, thus (G, p) is a flexible framework.

Finally, we show that p_t is $\mathcal{C}_n$-symmetric. If $v \in R_j^i \cap B_\ell^k$, then

$$\omega v \in \tau(\omega) R_j^i \cap \tau(\omega) B_\ell^k = R_j^{(i+1 \bmod n)} \cap B_\ell^{(k+1 \bmod n)}.$$

Hence, $\overline{a}(\omega v) = \tau(\omega)\overline{a}(v)$ and $\overline{b}(\omega v) = \tau(\omega)\overline{b}(v)$. The same equalities hold also if v belongs to a partially invariant component, since then ωv is also in a partially invariant component. Using commutativity of rotation matrices, we conclude the proof by (2):

$$p_t(\omega v) = \tau(\omega)\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\overline{a}(v) + \tau(\omega)\overline{b}(v) = \tau(\omega)p_t(v). \qquad \blacktriangleleft$$

▶ **Example 3.3.** By using the construction described in Theorem 3.2 we can construct the $\mathcal{C}_n$-symmetric flexible frameworks given in Figures 6, 7 and 8.

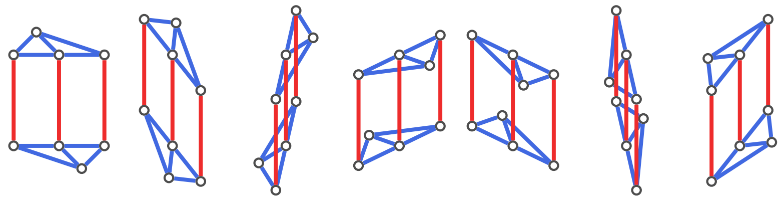

Figure 6 A flexible $\mathcal{C}_2$-symmetric placement for a given $\mathcal{C}_2$-symmetric NAC-colouring.

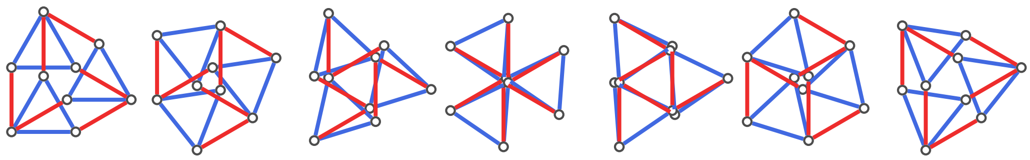

Figure 7 A flexible $\mathcal{C}_3$-symmetric placement for a given $\mathcal{C}_3$-symmetric NAC-colouring.

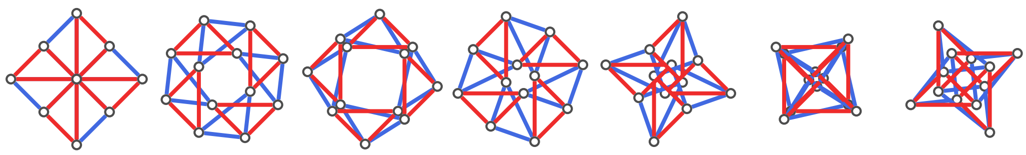

Figure 8 A flexible $\mathcal{C}_4$-symmetric placement for a given $\mathcal{C}_4$-symmetric NAC-colouring.

▶ **Example 3.4.** We consider the graph in Figure 9 with the given $\mathcal{C}_3$-symmetric NAC-colouring. Then the red 3-cycle forms a red partially invariant component. Therefore its position is fixed during the motion.

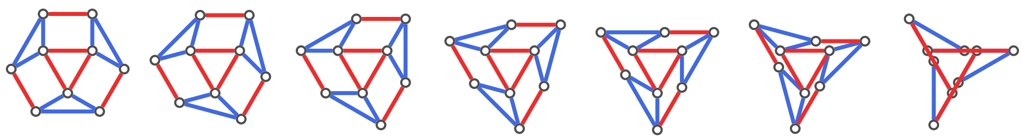

Figure 9 A flexible $\mathcal{C}_3$-symmetric placement for a given $\mathcal{C}_3$-symmetric NAC-colouring.

The constructed framework given by the proof of Theorem 3.2 may not be *proper flexible*, i.e., have no overlapping vertices, see Figure 10. As outlined in [8], there are necessary and sufficient conditions for determining when a graph will or will not have a proper flexible placements, although they have as of yet not been adapted fully to the symmetric case.

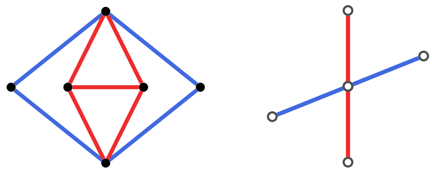

Figure 10 A flexible $\mathcal{C}_2$-symmetric placement for a given $\mathcal{C}_2$-symmetric NAC-colouring for which two vertices overlap.

▶ Remark. While we have only dealt with frameworks with rotational symmetry, there are other types of symmetry for the plane, namely reflectional and translational symmetry. Although flexible placements that preserve translational symmetry have very recently been investigated [2], not much is known for flexible placements that preserve reflectional symmetry or preserve both reflectional and rotational symmetry.

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