Monotone Arc Diagrams with few Biarcs*

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Abstract

We show that every planar graph can be represented by a monotone topological 2-page book embedding where at most $15n/16$ (of potentially $3n - 6$) edges cross the spine exactly once.

1 Introduction

Arc diagrams (Figure 1) are drawings of graphs that represent vertices as points on a horizontal line, called *spine*, and edges as *arcs*, consisting of a sequence of halfcircles centered on the spine. A *proper arc* consists of one single halfcircle. In *proper arc diagrams* all arcs are proper. In *plane* arc diagrams no two edges cross. Note that plane proper arc diagrams are also known as 2-*page book embeddings* in the literature. Bernhard and Kainen [2] characterized the graphs admitting plane proper arc diagrams: subhamiltonian planar graphs, i.e., subgraphs of planar graphs with a Hamiltonian cycle. In particular, non-Hamiltonian maximal planar graphs do not admit plane proper arc diagrams.

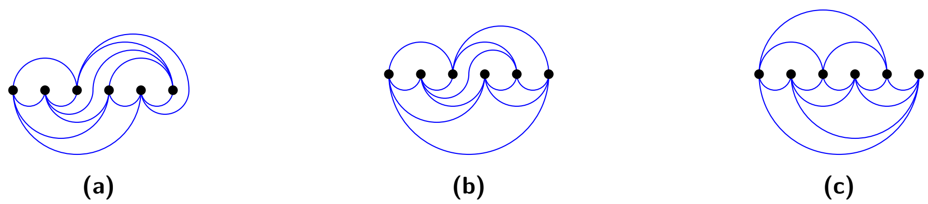

Figure 1 Arc diagram (a), monotone arc diagram (b), proper arc diagram (c) of the octahedron.

To represent all planar graphs, it suffices to allow each edge to cross the spine at most once [9]. The resulting arcs composed of two halfcircles are called *biarcs* (see Figure 1a). Additionally, all edges can be drawn as *monotone* curves w.r.t. the spine [6]; such a drawing is called a *monotone topological (2-page) book embedding*. A monotone biarc is either *down-up* or *up-down*, depending on if the left halfcircle is drawn above or below the spine, respectively. Note that a *monotone topological* 2-page book embedding is not necessarily a 2-page book embedding even though the terminology suggests it.

In general, biarcs are needed, but *some* edges can be drawn as proper arcs. Cardinal et al. [3] gave bounds on the required number of biarcs showing that every planar graph on

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This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

$n \geq 3$ vertices admits a plane arc diagram with at most $\lfloor (n-3)/2 \rfloor$ biarcs (not necessarily monotone). They also described a family of planar graphs on $n_i = 3i+8$ vertices that cannot be drawn as a plane biarc diagram using less than $(n_i - 8)/3$ biarcs for $i \in \mathbb{N}$. However, they use arbitrary biarcs. When requiring only monotone biarcs, Di Giacomo et al. [6] gave an algorithm to construct a monotone plane arc diagram that may create close to $2n$ biarcs for an n-vertex planar graph. Cardinal et al. [3] improved this bound to at most $n-4$ biarcs.

Results. As a main result, we improve the upper bound on the number of monotone biarcs:

▶ **Theorem 1.1.** *Every n-vertex planar graph admits a plane arc diagram with at most $\lfloor \frac{15}{16}n - \frac{5}{2} \rfloor$ biarcs that are all down-up monotone. Such a diagram is computable in $O(n)$ time.*

For general arc diagrams, $\lfloor (n-8)/3 \rfloor$ biarcs may be needed [3], but it is conceivable that this number increases for monotone biarcs. We investigated the lower bound with a SAT based approach (based on [1]), with the following partial result; details will appear in the full version.

▶ **Observation 1.2.** *Every Kleetope on $n' = 3n - 4$ vertices derived from triangulations of $n \leq 14$ vertices admits a plane arc diagram with $\lfloor (n'-8)/3 \rfloor$ monotone biarcs.*

Note that a *Kleetope* is derived from a planar triangulation T by inserting a new vertex v_f into each face f of T and then connecting v_f to the three vertices bounding f.

Related Work. Giordano et al. [8] showed that every upward planar graph admits an *upward topological book embedding* where edges are either proper arcs or biarcs. One of their directions for future work is to minimize the number of spine crossings. Note that these embeddings are monotone arc diagrams with at most one spine crossing per edge respecting the orientations of the edges. Everett et al. [7] used monotone arc diagrams with only down-up biarcs to construct small universal point sets for 1-bend drawings of planar graphs. This result was extended by Löffler and Tóth [10] by restricting the set of possible bend positions. They use monotone arc diagrams with at most $n-4$ biarcs to build universal points set of size $6n-10$ (vertices and bend points) for 1-bend drawings of planar graphs on n vertices. Using Theorem 1.1, we can slightly decrease the number of points by approximately $n/16$.

2 Overview of our Algorithm

To prove Theorem 1.1 we describe an algorithm to incrementally construct an arc diagram for a given planar graph G on n vertices. W.l.o.g. we assume that G is a (combinatorial) *triangulation*, i.e., a maximal planar graph. Our algorithm is a (substantial) refinement of the algorithm of Cardinal et al., which is based on the notion of a canonical ordering. A canonical ordering is defined for an *embedded* triangulation. Every triangulation on $n \geq 4$ vertices is 3-connected, so selecting one facial triangle as the *outer face* embeds it into the plane which determines a unique outer face (cycle) for every biconnected subgraph. A *canonical ordering* [5] of an embedded triangulation G is a total order of vertices $v_1, \ldots, v_n$ s.t.

- for each $i \in \{3, \ldots, n\}$, the induced subgraph $G_i = G[\{v_1, \ldots, v_i\}]$ is biconnected and internally triangulated (i.e., every inner face is a triangle);
- for each $i \in \{3, \ldots, n\}$, (v_1, v_2) is an edge of the outer face C_i of G_i;
- for each $i \in \{3, \ldots, n-1\}$, v_{i+1} lies in the interior of C_i and the neighbors of v_{i+1} in G_i form a sequence of consecutive vertices along the boundary of C_i.

Every triangulation admits a canonical ordering [5] and one can be computed in $O(n)$ time [4]. We say that a vertex v_i *covers* an edge e (a vertex v, resp.) if and only if e (v, resp.) is an edge (vertex, resp.) on C_{i-1} but not an edge (vertex, resp.) on C_i.

We iteratively process the vertices in a canonical order $v_1, \ldots, v_n$. Every vertex v_i arrives with α credits that we can either spend to create biarcs (at a cost of one credit per biarc) or distribute on edges of the outer face C_i for later use. We prove our claimed bound by showing that each biarc drawn can be paid for s.t. at least seven credits remain in total.

There are two types of proper arcs: *mountains* (above the spine) and *pockets* (below the spine). The following invariants hold after processing vertex v_i, for every $i \in \{3, \ldots, n\}$.

(I1) Every edge is either a proper arc or a down-up biarc.
(I2) Every edge of C_i is a proper arc. Vertex v_1 is the leftmost and v_2 is the rightmost vertex of G_i. Edge (v_1, v_2) forms the lower envelope of G_i, i.e., no point of the drawing is vertically below it. The other edges of C_i form the upper envelope of G_i.
(I3) Every mountain whose left endpoint is on C_i carries 1 credit.
(I4) Every pocket on C_i carries π credits, for some constant $\pi \in (0, 1)$.
(I5) Every biarc in G_i carries (that is, is paid for with) 1 credit.

Usually, we insert v_i between its leftmost neighbor ℓ_i and rightmost neighbor r_i along C_{i-1}. The algorithm of Cardinal et al. [3] gives a first upper bound on the insertion costs.

▶ **Lemma 2.1.** *If v_i covers at least one pocket, then we can insert v_i maintaining (I1) to (I5) using ≤ 1 credit. If $\deg_{G_i}(v_i) \geq 4$, then $1 - \pi$ credits are enough.*

Proof (Sketch). We place v_i in the rightmost covered pocket and pay for at most 1 mountain; see Figures 2a and 2b. If $\deg_{G_i}(v_i) \geq 4$, at least 1 covered pocket's credits is free. ◀

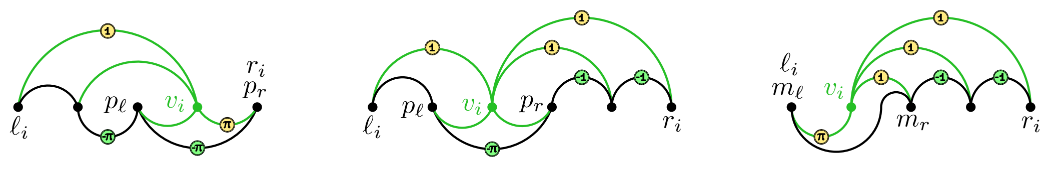


Figure 2 Inserting a vertex v_i using $1 - \pi$, 1, and $1 + \pi$ credits, resp. (Lemma 2.1–2.2).

▶ **Lemma 2.2.** *If v_i covers mountains only, then we can insert v_i maintaining (I1) to (I5) using $\leq 1 + \pi$ credits. If $\deg_{G_i}(v_i) \geq 4$, then $5 - \deg_{G_i}(v_i)$ credits suffice.*

Proof (Sketch). If $\deg_{G_i}(v_i) < 4$, we push down the leftmost mountain and place v_i on the created biarc paying for 1 mountain and 1 pocket each; see Figure 2c. If $\deg_{G_i}(v_i) \geq 4$, we push down the rightmost mountain saving the credit of a covered mountain; see Figure 3. ◀

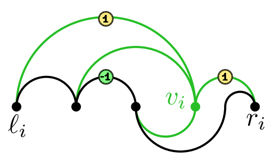


Figure 3 An alternative drawing to insert a degree four vertex.

Full proofs of Lemmas 2.1 and 2.2 will appear in the full version. We only steal credits from arcs on C_{i-1} in both proofs. If left endpoints of proper arcs not on C_{i-1} are covered, there is slack.

In the following, we prove that we can choose $\pi = 1/8$, so that to achieve the bound of Theorem 1.1, we insert a vertex at an average cost of $1-\pi/2$. Lemmas 2.1 and 2.2 guarantee this bound only in certain cases, e.g., a sequence of three degree two (in G_i) vertices stacked onto mountains costs $1+\pi$ per vertex and produces three biarcs, see Figure 4a. A symmetric scheme with up-down biarcs realizes the same graph with one biarc; see Figure 4b.

(a) **(b)**

Figure 4 A sequence of degree two vertices in forward (a) and reverse drawing (b).

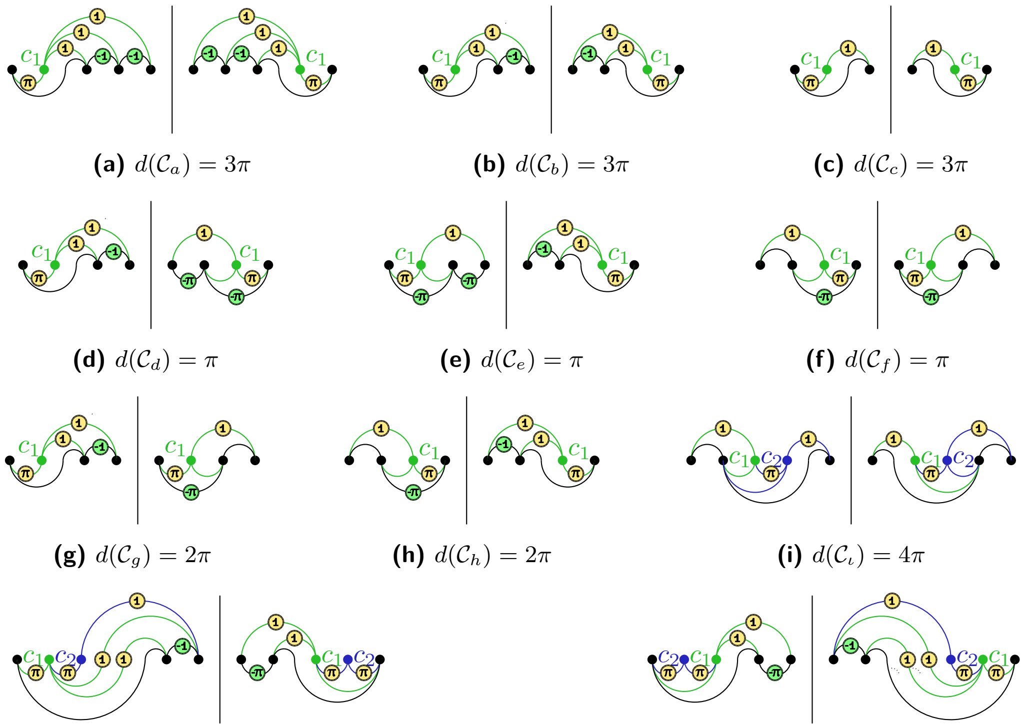
To exploit this behavior, we consider the instance in both a *forward drawing*, using only proper arcs and down-up biarcs, and a *reverse drawing* that uses only proper arcs and up-down biarcs (and so (I1) and (I3) appear in a symmetric formulation). Out of the two resulting arc diagrams, we choose one with a fewest number of biarcs. To prove Theorem 1.1, we need to insert a vertex at an average cost of $\alpha = 2 - \pi$ credits into both diagrams.

The outer face, a sequence of pockets and mountains, can evolve differently in both drawings because edges covered by a vertex may not be drawn the same way in both drawings. Further, it does not suffice to consider a single vertex in isolation. For instance, consider a degree three vertex inserted above two mountains in both the forward and reverse drawings; see Figure 5b. In each drawing, this costs $1 + \pi$ credits, or $2(1 + \pi)$ in total. W.r.t. our target value $\alpha = 2 - \pi$, these costs incur a *debt* of 3π credits. Indeed, there are several such *open configurations*, listed in Figure 5, for which our basic analysis does not suffice.

Each open configuration $\mathcal{C}$ consists of up to two adjacent vertices on the outer face whose insertion incurred a debt and their incident edges. It specifies the drawing of these edges, as pocket, mountain, or biarc in forward and in reverse drawing, as well as the drawing of the edges covered by the vertices of the open configuration. When a vertex v_i covers (part of) an open configuration, we may alter the placement of the vertices and/or draw the edges of the open configuration differently. The associated debt $d(\mathcal{C})$ is the amount of credits paid in addition to α credits per vertex. As soon as any arc of an open configuration is covered, the debt must be paid or transferred to a new open configuration. We enhance our collection of invariants as follows.

(I6) A sequence of consecutive arcs on C_i may be associated with a debt. Each arc is part of at most one open configuration; refer to Figure 5 for a full list of such configurations.

To prove Theorem 1.1 we show that the credit total carried by arcs in both drawings minus the total debt of all open configurations does not exceed $\alpha i - 5$ after inserting v_i.



(a) $d(\mathcal{C}_a) = 3\pi$ **(b)** $d(\mathcal{C}_b) = 3\pi$ **(c)** $d(\mathcal{C}_c) = 3\pi$

(d) $d(\mathcal{C}_d) = \pi$ **(e)** $d(\mathcal{C}_e) = \pi$ **(f)** $d(\mathcal{C}_f) = \pi$

(g) $d(\mathcal{C}_g) = 2\pi$ **(h)** $d(\mathcal{C}_h) = 2\pi$ **(i)** $d(\mathcal{C}_\iota) = 4\pi$

(j) $d(\mathcal{C}_j) = 5\pi$ **(k)** $d(\mathcal{C}_k) = 5\pi$

Figure 5 The set of open configurations. Each subfigure shows the forward drawing (left) and the reverse drawing (right) and is captioned by the debt incurred.

3 Default insertion of a vertex v_i

If v_i does not cover any arc of an open configuration, we use procedures from Lemmas 2.1 and 2.2. If $\deg_{G_i}(v_i) \geq 4$ and v_i covers any pocket in either drawing, by Lemmas 2.1 and 2.2 the insertion costs are at most $2 - \pi = \alpha$. If $\deg_{G_i}(v_i) \geq 5$ and v_i only covers mountains in both drawings, the costs are 0 (Lemma 2.2). If $\deg_{G_i}(v_i) = 4$ and v_i covers mountains only in both drawings, we obtain the open configuration in Figure 5a with cost $2+2\pi$ and debt 3π.

If $\deg_{G_i}(v_i) = 2$ and v_i covers a pocket in one drawing, insertion in this drawing costs π resulting in total cost $\leq 1 + 2\pi$ or at most α if $\pi \leq 1/3$. If $\deg_{G_i}(v_i) = 2$ and v_i covers only mountains, we have the open configuration in Figure 5c with cost $2 + 2\pi$ and debt 3π.

It remains to consider $\deg_{G_i}(v_i) = 3$. There are four pocket-mountain configurations for two arcs of G_{i-1} covered by v_i: MM, MP, PM, and PP (using M for mountain and P for pocket). Pattern PP costs $1 - \pi$, pattern MM costs $1 + \pi$. Each drawing has its favorite mixed pattern (PM for forward and MP for reverse) with cost 0; the other pattern costs 1.

There is only one *forward*|*reverse* combination, $MM|MM$, with cost $2+2\pi$ and debt 3π, leading to the open configuration in Figure 5b. Two combinations, $MM|PM$ and $MP|MM$, have cost $2+\pi$ and debt 2π resulting in open configurations in Figure 5g and 5h, resp. Also, the combinations $MM|PP$, $PP|MM$, and $MP|PM$ with costs 2 and a debt π lead to open configurations in Figure 5d, 5e, and 5f, resp. All other combinations cost at most α.

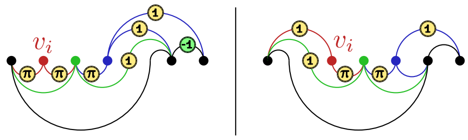


Figure 6 Alternative drawing to handle an open configuration $\mathcal{C}_\iota$ for $\deg_{G_i}(v_i) = 2$.

4 When and how to pay your debts

In this section, we describe the insertion of v_i if it covers an arc of an open configuration. Note that (1) every open configuration contains at least one mountain and at least one pocket in both drawings; (2) the largest debt incurred by one open configuration is 5π. Open configurations $\mathcal{C}_\iota, \mathcal{C}_j, \mathcal{C}_k$ (with highest debts) are introduced in the discussion below.

Case 1: $\deg_{G_i}(v_i) = 2$**.** If v_i covers a pocket of an open configuration $\mathcal{C}$ in either drawing, the insertion costs of $\pi + (1 + \pi)$ cover $d(\mathcal{C})$, as long as $1 + 2\pi + 5\pi \le \alpha$, that is, $\pi \le 1/8$. Assume v_i covers a mountain of open configuration $\mathcal{C}$ in both drawings; i.e., $\mathcal{C} \in \{\mathcal{C}_g, \mathcal{C}_h, \mathcal{C}_\iota\}$. If $\mathcal{C} \in \{\mathcal{C}_g, \mathcal{C}_h\}$, we obtain the open configurations in Figure 5j and 5k, resp., with cost $4 + 3\pi$ (for both vertices) and debt 5π. Otherwise $\mathcal{C} = \mathcal{C}_\iota$, and we use the drawings shown in Figure 6 (where v_i is inserted on the left mountain; the other case is symmetric). The costs are $2 + 3\pi$ (forward) and $3 + 2\pi$ (reverse), totaling $5 + 5\pi \le 3\alpha$, for $\pi \le 1/8$.

Case 2: $\deg_{G_i}(v_i) \ge 5$ **and Case 3:** $\deg_{G_i}(v_i) \in \{3, 4\}$**.** In the full version, we will discuss both cases in detail while we only mention the main ideas here. Each open configuration includes a mountain that can pay the debt if the configuration is entirely covered. We only focus on the left- and rightmost open configurations $\mathcal{C}_\ell$ and $\mathcal{C}_r$. In both cases, we mainly carefully move vertices v_i, c_1 and c_2 of $\mathcal{C}_r$ to avoid covered mountains from becoming biarcs, i.e., saving their credits.

5 Summary & Conclusions

Proof of Theorem 1.1. As previously shown, if $\pi \le 1/8$, we maintain all invariants with α credits per vertex. G_3 is a triangle with two pockets on C_3 in both orientations, i.e. G_3 costs 4π. As v_1,v_2 and v_3 contribute 3α credits, there are $6 - 7\pi > 5$ unused credits after drawing G_3. If there remains an open configuration in G_n, there is a mountain with a credit paying its debt. Hence, the 5 unused credits of G_3's drawing remain. As a canonical ordering is computable in $O(n)$ time and we backtrack $O(1)$ steps if needed, the runtime follows. ◀

We proved the first upper bound of the form $c \cdot n$, with $c < 1$, for the total number of monotone biarcs in arc diagrams of n-vertex planar graphs. In our analysis, only three subcases require $\pi \le 1/8$, i.e., a refinement may provide a better upper bound. Also, it remains open if there is a planar graph that requires more biarcs in a monotone arc diagram than in a general arc diagram. Finally, narrowing the gap between lower $\lfloor \frac{n-8}{3} \rfloor$ and upper $\lfloor \frac{15}{16}n - \frac{5}{2} \rfloor$ bounds would be interesting, particularly from the lower bound side.

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