Colouring bottomless rectangles and arborescences

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Abstract

We study problems related to colouring bottomless rectangles. We show that there is no number m and semi-online algorithm to colour a family of nested bottomless rectangles from below with a bounded number of colours such that every m-fold covered point is covered by at least two colours. We also prove several similar results that follow from a more abstract arborescence colouring problem, which is interesting on its own. Our key result is that there is no semi-online algorithm to colour the vertices of an arborescence without producing a long monochromatic path when the vertices are presented in a leaf-to-root order. The lower bounds are complemented with simple optimal upper bounds for semi-online algorithms from other directions.

1 Introduction

The systematic study of polychromatic colourings and cover-decomposition of geometric ranges was initiated by Pach over 30 years ago [5, 6]. The field has gained popularity in the new millennium, with several breakthrough results; for a (slightly outdated) survey, see [7], or see the up-to-date interactive webpage http://coge.elte.hu/cogezoo.html (maintained by Keszegh and the first author).

Our paper focuses on the colouring of one particular geometric family, known as *bottomless rectangles*. A subset of $\mathbb{R}^2$ is called a (closed) bottomless rectangle if it consists of the points $\{(x, y) \mid l \leq x \leq r, y \leq t\}$ for some parameters (l, r, t). These range spaces were first defined by Asinowski et al. [1], who showed that for any positive integer k, any finite set of points in $\mathbb{R}^2$ can be k-coloured such that any bottomless rectangle with at least $3k-2$ points contains all k colors. They also showed that the optimal number that can be written in place of $3k-2$ in the above statement is at least $1.67k$. Their upper bound giving $3k-2$ is a very neat *semi-online algorithm*.

Our paper studies the dual of the above problem. Our goal is to find the optimal m_k for which any finite collection of bottomless rectangles can be k-colored such that any m_k-fold covered region is covered by all k colors. About this question much less is known; the best upper bound $m_k = O(k^{5.09})$ is a corollary of a more general result [2] about *octants* (combined with an improvement of the base case [4] that slightly lowered the exponent). The general conjecture, however, is that $m_k = O(k)$ for any family [7]. It was also proved in [2] that there is no semi-online algorithm "from above" for colouring bottomless rectangles. An algorithm is said to colour bottomless rectangles "from above" if it is presented the rectangles in decreasing order of height; "from below" is defined analagously. Similarly, an algorithm is said to colour bottomless rectangles "from the left" if it is presented the rectangles in increasing order of *left* endpoint ($\rightarrow$), and "from the right" if it is presented the rectangles in decreasing order of *right* endpoint ($\leftarrow$). A *semi-online algorithm* is one where the vertices

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are presented in some order, and the algorithm need not colour a vertex as soon as it is presented, but may not recolour previously coloured vertices. This is a natural generalisation of *online* algorithms, which must colour vertices as soon as they are presented.

Our main result for bottomless rectangles is a generalisation of the non-existence of a semi-online algorithm from above.

▶ **Theorem 1.1.** *For any numbers k and m, for any semi-online algorithm that k-colours bottomless rectangles from above, below, the left, or the right, there is a family of bottomless rectangles that the algorithm colours such that there will be a m-fold covered point that is covered by at most one colour.*
Moreover, the family of the bottomless rectangles can be such that the boundaries of the rectangles are pairwise disjoint.

These are complemented by positive results, where we show that for each of four "natural" bottomless rectangle configurations there is a direction from which there is an optimal online algorithm. Our proof is much more complicated than the one in [2]; while they use an Erdős-Szekeres type incremental argument [3], we need a certain diagonalisation method. In particular, we reduce the semi-online bottomless rectangle colouring problem to a question about semi-online colourings of arborescences, which is interesting in its own right.

▶ **Theorem 1.2.** *For any numbers k and m, and any semi-online algorithm that k-colours the vertices of an arborescence in a leaf-to-root order, there is an arborescence such that the algorithm produces a directed path on m vertices that contains at most one colour.*

In Section 2 we present our main result, and in Section 3 we present our results on polychromatic colourings of bottomless rectangles. The full proofs and other related results can be found in the full version of our paper available at `https://arxiv.org/abs/1912.05251`.

2 Arborescences

An *arborescence* is a directed tree with a distinguished vertex called a *root* such that all the edges are directed away from the root, i.e., there is exactly one directed path to any vertex from the root. (See Figure 1.) A disjoint union of arborescences is an *arborescence forest*, also called a *branching*. We say that an ordering of the vertices of a branching is *root-to-leaf* if every vertex is preceded by its in-neighbors and succeeded by its out-neighbors; in particular, from every component first the root is presented and last a leaf. The following claim easily generalises the notion of a proper colouring for arborescences.

▶ Claim 2.1. The vertices of any arborescence can be k-coloured by an online algorithm in a root-to-leaf order such that any directed path on k vertices contains all k colours.

We call the reversal of a root-to-leaf ordering a *leaf-to-root* ordering; in particular, from every component first a leaf is presented and last the root.

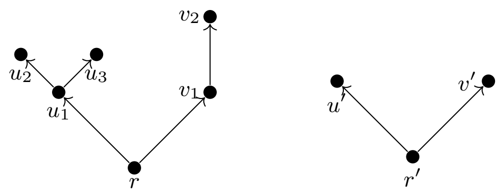

Figure 1 A branching with roots r and r'. A leaf-to-root ordering might present the vertices u', v' and r' before u_3, so it is not necessary that the roots of the branching are the last vertices presented.

Our main result, Theorem 1.2, shows that a similar semi-online polychromatic k-colouring algorithm that takes the vertices in a leaf-to-root order cannot exist, moreover, any such algorithm will even leave an arbitrarily long path monochromatic. To be able to apply this result for bottomless rectangles, we will need (and prove) a slightly stronger notion.

Denote the roots of the branching before a new vertex u is presented by $v_1, v_2, \dots$ indexed in the order in which they were presented. We say that a *leaf-to-root* ordering is *geometric* if the parents of u form an interval in this order, i.e., for every u, $\{v_i \mid u \lessdot v_i\} = \{v_i \mid l < i < r\}$ for some l and r.

▶ **Theorem 2.2.** *There is no semi-online k-colouring algorithm that receives the vertices of an arborescence in a geometric leaf-to-root order, and maintains at any stage (without recolouring any vertices) that all directed paths on m vertices contain at least two colours.*

The key idea of the proof is to exploit that any path of length m must contain at least 2 colours. Further, when a new vertex p is presented, we only need to consider paths of length m rooted at p to check that the colouring is proper. These two conditions show that the algorithm can produce "essentially" only finitely many colourings. More precisely, when p is presented, we "trim" the arborescence of depth m rooted at p to remove any repetitions (see Figure 2 for an example). After this trimming process, we are left with only finitely many different trees. Then we assume that an algorithm has already forced all "achievable" isomorphic trees to appear, and present a new vertex below the root of each. The newly obtained tree must be isomorphic to one of its parents', which leads to a contradiction. For the detailed proof, see the full version of the paper.

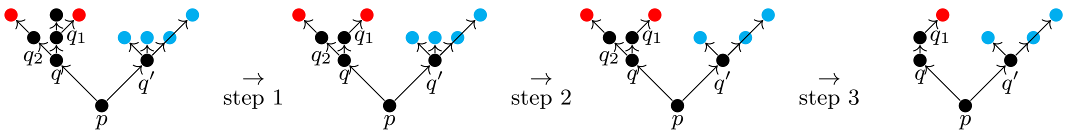

Figure 2 Example for trimming with $m = k = 2$. In step 1, we delete uncoloured points at distance > 2 from p. In step 2, we "trim" the repeated blue parents of q'. In step 3, the subtrees rooted at q_1 and q_2 are isomorphic, so we delete q_2. After this process, we have not lost any information about the paths of length 2 rooted at p.

3 Bottomless rectangles

3.1 Bottomless rectangle configurations

Using the classical result of Erdős and Szekeres [3] that any length $(r-1)(s-1)+1$ sequence of numbers contains either an increasing subsequence of length r or a decreasing subsequence of length s, we will define four configurations of bottomless rectangles as follows.

Ordering the rectangles first by left, then by right endpoints, we obtain that any point contained in $(m-1)^4+1$ rectangles is contained in m bottomless rectangles in an *Erdős Szekeres configuration.* We name these configurations *increasing/decreasing steps, towers* and *nested rectangles.*

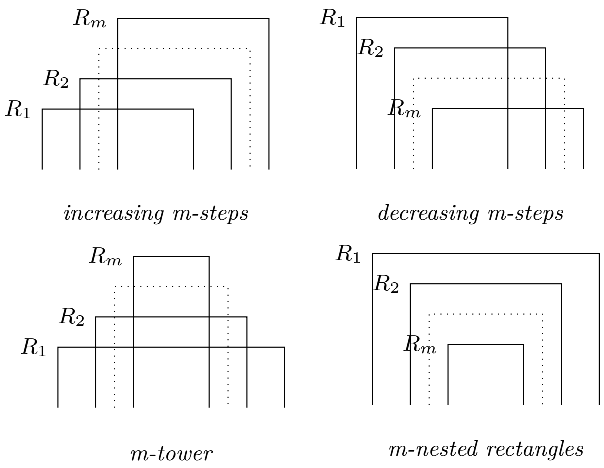

Figure 3 Erdős-Szekeres configurations.

Restricting the colouring problem, we obtain that $m_k = k$ for each fixed configuration. That is, for example, any family of bottomless rectangles can be k-coloured with an online algorithm from the right so that any point covered by increasing k-steps is covered by all k colours. Although these configurations have a seemingly simple structure, we apply the arborescence colouring problem to show that the existence of a semi-online colouring algorithm depends on the order in which the rectangles are coloured.

For a full summary, see the table below.

	left($\rightarrow$)	right($\leftarrow$)	below($\uparrow$)	above($\downarrow$)
inc. steps	∞	$= k$	∞	$= k$
dec. steps	$= k$	∞	∞	$= k$
towers	$= k$	$= k$	$= k$	∞
nested	$= k$	$= k$	∞	$= k$

3.2 Colouring algorithms

▶ **Corollary 3.1.** *There is no semi-online colouring algorithm for towers from above, i.e., for any numbers k and m, for any semi-online algorithm that k-colours bottomless rectangles from above, there is a family of bottomless rectangles such that any two intersecting rectangles form a tower, and the algorithm colours them such that there will be a m-fold covered point that is covered by at most one colour.*

In order to apply Theorem 2.2, we need to show that any branching can be realised as a family of towers so that

1. ordering the rectangles from above corresponds to a geometric leaf-to-root order of the branching, and
2. a semi-online colouring algorithm for towers from above corresponds to an appropriate semi-online k-colouring algorithm for branchings in this order.

We construct this realisation by induction on the number of vertices of the branching. We will need the geometric property of the leaf-to-root ordering to ensure that each isolated vertex is realised as a disjoint rectangle to the right of the former ones. Thus, when a new vertex s is presented, its parents will correspond to some geometrically adjacent rectangles, and s can be realised as a rectangle in the desired configuration.

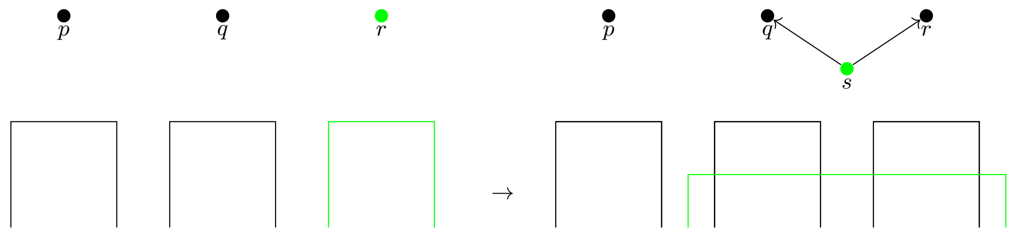

Figure 4 Each time a disjoint element (such as r) is presented, we realise it as a disjoint rectangle to the right. We are then able to realise s as a minimal element that forms a tower with q and r. We need the ordering to be geometric so that the parents of s are consecutive rectangles.

An analagous construction with nested rectangles shows that the same statement holds for nested rectangles from below.

▶ **Corollary 3.2.** *There is no semi-online k-colouring algorithm from the left or from below for increasing steps. More precisely, for any integers k and m, there is no semi-online algorithm to k-colour rectangles from the left (or from below) so that at every step, any point covered by m-increasing steps is covered by at least 2 colours. Similarly, there is no semi-online colouring algorithm for decreasing steps from the right or from below.*

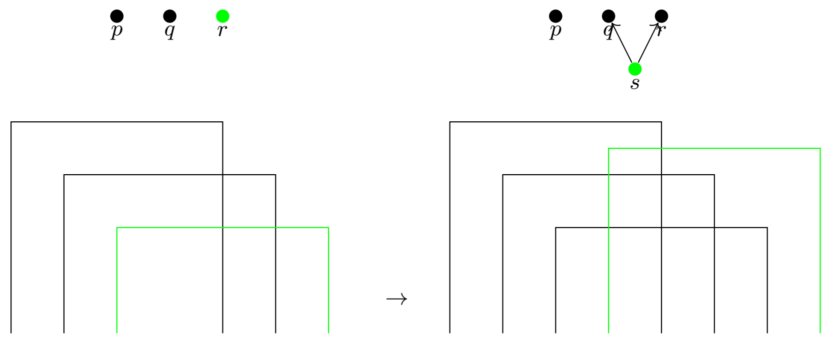

Figure 5 When a disjoint element (r) is presented, we realise it in decreasing steps with the other minimal elements. We are then able to realise s in increasing steps with q and r (once again relying on the geometric ordering).

Note that this statement is slightly weaker than Theorem 1.1 or Corollary 3.1 because we do not exclude the other kind of configurations from the family. Figure 5 shows the modification of this construction for increasing steps from the left.

For online algorithms from other directions, we can prove an optimal upper bound.

▶ **Theorem 3.3.** *Each configuration can be k-coloured so that $m_k = k$.*

These algorithms are simple compared to the proof of non-existence of algorithms from other directions. For example, when colouring with respect to k-towers from below, the algorithm proceeds as follows. When we present a rectangle R_t, define y_i to be maximal so that any point in R_t below y_i is already covered by colour i (and to be $-\infty$ if this does not exist). Then if $y_1 \geq y_2 \geq \dots y_k$, we colour R_t with colour k. The algorithms for the other configurations are analagous, as depicted in Figures 6 and 7, while the detailed proofs can again be found in the full version.

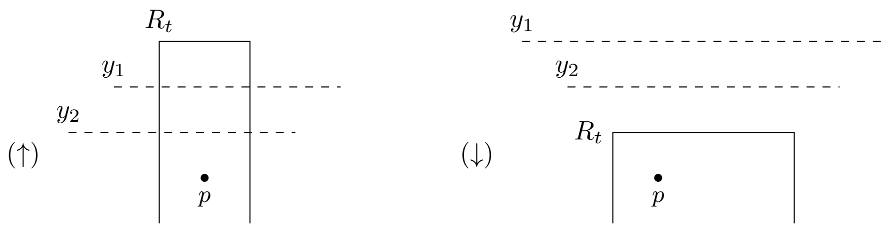

Figure 6 Colouring algorithms for k-towers and k-nested sets, respectively.

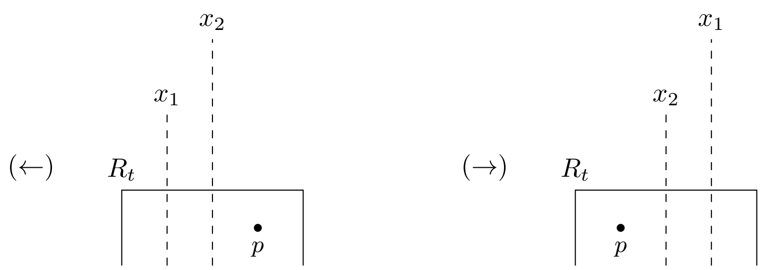

Figure 7 Colouring algorithms for increasing and decreasing k-steps, respectively.

3.3 An improved lower bound

Finally, we prove the following lower bound for general bottomless rectangle families.

▶ **Theorem 3.4.** *$m_k \geq 2k - 1$ for bottomless rectangles, i.e., for any k there is a family of bottomless rectangles such that for every k-colouring of the family there is a point of the plane that is contained in at most $k - 1$ of the colors, although it is covered by $2k - 2$ rectangles.*

Our lower bound construction proceeds in two steps.

1. If $m_k < m_{k-1} + 2$, then every family has a polychromatic k-colouring that is proper (see Figure 8).
2. There is a family so that no polychromatic k-colouring is proper (see Figure 9).

This contradiction shows that $m_k \geq m_{k-1} + 2$, so by induction $m_k \geq 2k - 1$. Again, for the details see the full paper.

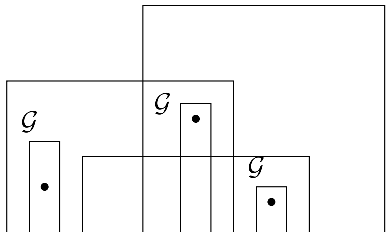

Figure 8 If no polychromatic colouring of $\mathcal{F}$ is proper, place disjoint thin copies of a test family $\mathcal{G}$ around every 2-covered point in $\mathcal{F}$ to obtain a contradiction.

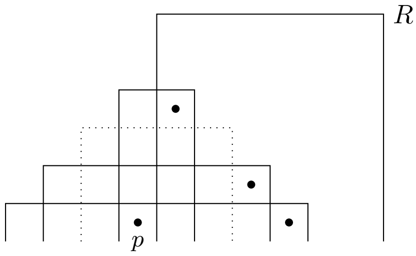

Figure 9 No polychromatic colouring of this family will be proper.

3.4 Concluding remarks

To summarise, our main result shows that by considering arborescences instead of hypergraphs associated to bottomless rectangles, there is no semi-online algorithm to properly colour bottomless rectangles from any of the four "natural" directions. In fact, with a slight modification, we can show that there is no semi-online algorithm to properly colour bottomless rectangles from any other direction either (i.e. along a line).

However, since online algorithms show that $m_k = k$ for each fixed configuration, the next natural step is to attempt to combine these colourings for general families. The strongest such result we have been able to prove is that if a family of bottomless rectangles contains no towers, then it can be k-coloured so that $m_k = O(k^2)$. The crux of the proof is to exploit that (1) adding nested sets to a family can increase m_k by a factor of at most k, and (2) colouring steps with respect to points turns out to be a special case of the primal problem - colouring points with respect to bottomless rectangles. For more details on these and other results, we again refer to the full version of the paper on arXiv.

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