Counting and Enumeration in Geometry, March 20, 2020
Exercises with Solutions

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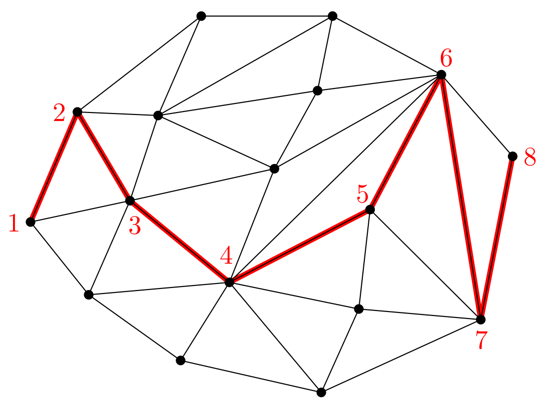
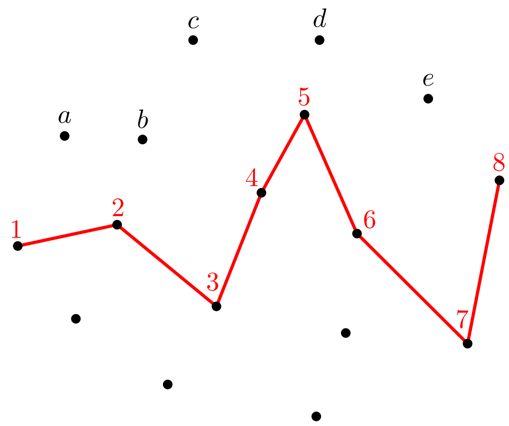
1. Sweeping a rope over a triangulation (Alvarez and Seidel, 2013)

   (a) What was the last new edge or pair of consecutive edges of the rope in the triangulation on the left? What is the next rope?

   (b) Which edge or pair of consecutive edges could have been the last added items, for some triangulation of the point set on the right for which the shown rope appears in the sweep?

   (c) For each resulting marked rope, determine the marked successor ropes.





Solutions: a) The last new pair of edges was 567. The next rope will shortcut 456 to 45.

b) The edges 34, 45, 67, and 78 are excluded, because there is no point below them with which they could form the last triangle of the sweep. The edge pairs 234, 567, and 678 are concave, so they cannot be the upper boundary of a triangle.

The remaining potential edges and edge pairs are 12, 23, 56, 123, 345, 456. (I am not sure that they all can really arise during the sweep.)

c) *Corrected solution. In the original version of the slides during the talk, and the first version of this solution, changes to the* right *of the mark were forbidden. The correct procedure is to put the mark at the* left *boundary of the changed edges, and to forbid changes that are completely to the left of the mark.*

The mark can be on 1, 2, 3, 4, or 5.

For 5, we have four possibilities: We can add the point d, with the edges $5d6$ and the mark at 5. We can shortcut 567 to 57, leaving the mark at 5. We can shortcut 678 to 68 with the mark at 6. We can replace 67 by $6e7$ with the mark at 6.

For 4, we have the same four possibilities.

For 3, we have the additional option of shortcutting 234 to 24, with the mark at 2.

For 2, there are two additional choices: add point b or point c, with the mark at 2.

For 1, we have one more choice: add the point a with the edges $1a2$ and the mark at 1.

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2. Bipolar orientations

   A bipolar orientation is a planar DAG (directed acyclic graph) with a single source (vertex without incoming edges) and a single sink (vertex without outgoing edges), drawn in the plane such that the both the source and the sink lie on the outer face.

   (a) Prove that around every vertex v except the source and the sink, the outgoing edges form a connected subsequence in the cyclic order around v.

   (b) Prove that in the clockwise order around each face, the edges of the face cycle can be partitioned into a contiguous subsequence of forward (clockwise) edges and a contiguous subsequence of backward (counterclockwise) edges.

Solutions: a) Suppose there are four incident vertices a, b, c, d in cyclic order such that the directed arcs are av, vb, cv, vd. Extend the arcs vb and vd to paths to the sink The forward paths must eventually meet, and form an undirected cycle K through bvd. The arcs av and cv point into different regions of K, one insided the cycle and one outside. Thus if we extend these arcs backwards to the source, one of the backward paths must intersect K, and therefore one of the two forward paths. This creates a cycle in the graph.

b) This is analogous to (a) for the dual graph. But let us prove it without the knowledge that the dual graph is a bipolar orientation, by counting pairs of neighboring edges incident to a common vertex v that have different orientations: one directed towards v and the other directed away from v. By (a), the number of such "alternations" is $2(n-2)$. When looking at a boundary cycle of d edges around a face F, it is clear that there are at most $d-2$ such alternations, because otherwise it would form a directed cycle. (The number of non-alternations must be even.) Adding up all faces F, we get at most

$$\sum_F (d_F - 2) = 2m - 2|F| = 2(n-2)$$

alternations in total, by Euler's formula. (It is easy to see that a bipolar orientation must be connected, since a DAG must contain at least one source and one sink per connected component.) This is the number that we calculated above. It follows that every face cycle has exactly $d-2$ alternations, and hence it looks as claimed.

The argument works also for the case that the graph is not biconnected, and a face cycle visits two sides of the same edge. (As an additional exercise, you may show that in a bipolar orientation, this situation can only happen for the outer face.)

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3. Vertical order among convex regions (Guibas and Yao, 1980)

   We have a finite set of disjoint convex polygons.

   (a) Among all polygons whose rightmost point p is visible from above (in the sense that the vertical upward ray from p is disjoint from all polygons), choose the polygon P whose rightmost point p is leftmost. Prove that P can be translated vertically upward to infinity without colliding with the other polygons.

   (b) We say that P is *below* Q if there are points $p \in P$ and $q \in Q$ on the same vertical line with p below q. Show that this relation has no cycles.

   (c) What is the situation in three dimensions?

Solutions. a) If P would hit another polygon Q, consider the polygons Q that it would hit (considering only P and Q in isolation). The rightmost point q of each such polygon Q must lie above P and to the left of p, because otherwise Q would intersect the vertical upward ray from p. (Here we use convexity.) Among all these polygons Q, take the one whose rightmost point q is *rightmost*. If q is not visible from above, the upward ray from q would intersect another polygon Q' that would also block the movement of P and whose rightmost point q' lies to the right of q. Thus, q is visible from above and left of p, contradicting the choice of P. (The argument would have to be refined to cover degenerate situations, if there are several rightmost points with the same x-coordinate.)

b) By (a), we can find a "maximal" element in the relation: a polygon that is below no other. Remove it and repeat. (In (a) and (b), it is sufficient to assume that the polygons are x-monotone (and connected).)

c) In space, the situation is different. It is easy to put three convex sticks on a table so that A lies above B in the sense of (b), B lies above C, and C lies above A.

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4. Sweeping a rope over a line arrangement

   We have n non-vertical lines, no two of which are parallel. We want to sweep a rope across the arrangement of lines, starting from the boundary of the bottom face and ending at the boundary of the top face, flipping it over a single face at a time.

   Prove that this can be done with a rope that has always at most $O(n)$ edges.

   (Experiments indicate that the true maximum is $2n-2$, including the two unbounded rays that every rope contains.)

Proof sketch: In a line arrangement, every vertex has two incoming and two outgoing edges. Look at the general situation (slide 15–3, slide 63 of 99), and consider the part of the red rope to the left of the crossing point. Let us look at the segments of the rope from left to right.

After a line a contributes an edge to the left part of the rope, it can either continue by crossing the blue rope, or it can disappear from the rope by "diving under" the red rope. If it crosses the blue rope, it cannot come back because all red lines cross the blue rope in the same direction. If a dives under the red rope, it runs from then on below the line b that it crosses. In this case it cannot reappear at the rope before b crosses the blue rope (and disappears), because a and b can cross only once. In this way, we can charge every vertex on the rope to a line that crosses the blue rope. This implies that there cannot be more than $O(n)$ edges.

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5. Matchings in a convex chain

   Let $S = (P_1, \dots, P_n)$ be a sequence of points in convex position. For $i = 0, 1, \dots, n$, let ℓ_i be a line that separates $P_1, \dots, P_i$ from $P_{i+1}, \dots, P_n$. For a matching M of S, we denote by M_i the part of M to the left of ℓ_i (on the side of the points $P_1, \dots, P_i$). In such a "partial matching", edges of S that cross ℓ_i appear as dangling edges ("half-edges"): Only their left endpoint is determined. Let B_i^k denote the number of partial matchings M_i with k dangling edges.

   (a) Find a recursion that computes the numbers $B_{i+1}^0, B_{i+1}^1, \dots$ from $B_i^0, B_i^1, \dots$.

   (b) Show that the matchings of S are in one-to-one correspondence with the so-called *Motzkin paths*: A Motzkin path of length n is a path from $(0,0)$ to $(n,0)$ that uses steps of the form $(1,-1)$, $(1,0)$ and $(1,1)$ and never goes below the x-axis.

Solutions: a) This is a much simpler case than the recursion for 3-chains without corners presented in the lecture:

$B_{i+1}^k = B_i^{k-1} + B_i^k + B_i^{k+1}$ for $k \geq 1$, and $B_{i+1}^0 = B_i^0 + B_i^1$.

b) The number of Motzkin paths that end at the point (i, k) satisfy the same recursion and the same starting conditions, and the one-to-one correspondence is obvious. The matchings in n points in convex position correspond to Motzkin paths of length n. The number of Motzkin paths of length n is called the n-th Motzkin number[1] M_n. It is known that $M_n = \Theta(3^n/n^{3/2})$.

6. Counting perfect matchings of point sets (Wettstein 2014)

   We sweep a rope over a perfect matching of a points set by adding the leftmost matching edge e for which every matching edge that lies below e (in the sense of exercise 3b) is already on or below the rope.

   We mark the left endpoint of e on the rope.

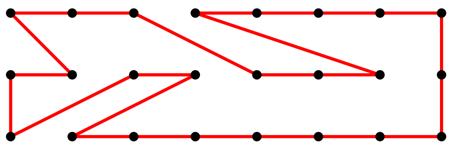
   Show that the number of perfect matchings of a point set can be determined in $O(n^3 2^n)$ time and $O(n 2^n)$ space.

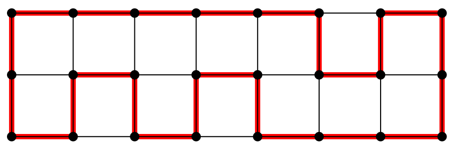
Solution sketch: The nodes of the DAG are the same as for triangulations: x-monotone ropes with one marked vertex. Every node has $O(n^2)$ outgoing edges, because every potential next matching edge is a segment between two points above the rope. Thus the number of arcs is $O(n^3 2^n)$, and this determines the running time.

The following exercises from an early draft were abandoned because they were deemed too difficult.

7. Hamilton cycles

   (a) Find a set of recursion formulas for counting the number H_n of noncrossing Hamilton cycles of the $3 \times n$ grid point set, like the one shown in the left figure for $n = 8$.





   (b) Determine or estimate the growth rate $\lim \sqrt[n]{H_n}$, possibly with the help of a computer.

   (c) If you are familiar with generating functions, you may try to derive a more precise asymptotic formula for H_n.

8. Hamilton paths

   If the previous exercise was too hard, do it for the $2 \times n$ grid, or for the $3 \times n$ grid *graph*, as shown in the right figure. If it was too easy, repeat it for Hamilton *paths*.

The solution for the $3 \times n$ grid graph was developed during the exercise session: This graph has $2^{n/2-1}$ Hamilton paths, for even n: The path has $n/2 - 1$ U-shaped indentations, and each indentation can be either from above or from below.

[1] `https://en.wikipedia.org/wiki/Motzkin_number`